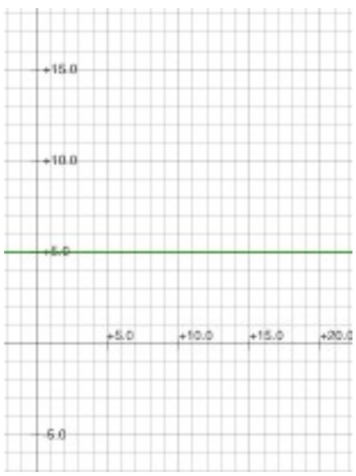
Mathematics Tutorial Series
Differential Calculus #6

Differentiation Rules I: Linear and Product Rules

Rule 1: Constants

If $y =$ (constant) then y never changes and its rate of change is 0.

So if k is a constant then $\frac{dk}{dx} = 0$.



Rule 2: Linear

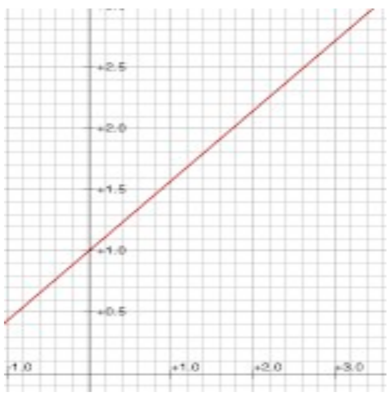
If $y = 17\ x$ then $\frac{dy}{dx} = 17$.

This is a special case of: $y =$ (constant) x

Here if $y = mx$ then $\frac{dy}{dx} = m$.

Or if $y\ =\ mx\ +\ b$ then $\frac{dy}{dx} = m$ also.

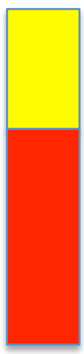
This is just a straight line.
It is the same as its tangent and has slope m.



Rule 3: Adding and subtracting

Suppose that $y = f(x) + g(x)$ is a sum of two functions. Rates of change simply add.

Think of this as a stack. Here yellow is graduate enrolment and red is undergraduate enrolment. The university enrolment grows (or shrinks) at a rate that is the sum of graduate growth and undergraduate growth.



$$\frac{dy}{dx} = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$

Example: Let $y = x^2 + 7x - 113$.

Then $\frac{dy}{dx} = 2x + 7$

Here is an outline of the proof:

$$\lim_{x \to a} \frac{\big(f(x) + g(x)\big) - \big(f(a) + g(a)\big)}{x - a}$$

$$\lim_{x \to a} \left(\frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}\right)$$

$$f'(a) + g'(a)$$

Rule 4: Products

Consider a function that is a product. Take $y = f(x)g(x)$.

So y is just $f(x)$ multiplied by $g(x)$.

How do we find the derivative of this product?

Think about an area with length L and width W.

The area is $A = L \times W$.

If both L and W are changing how does the area change?

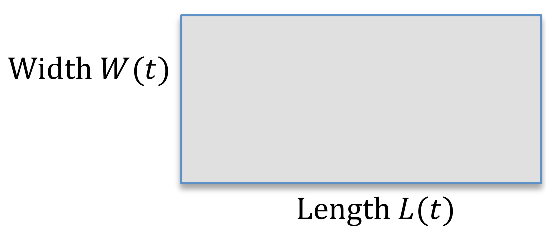
So what is the derivative of $A(t) = L(t)W(t)$?

The Product Rule says:

$$\frac{d\,A(t)}{dt} = \frac{dL(t)}{dt}\,W(t) + L(t)\frac{dW(t)}{dt}$$

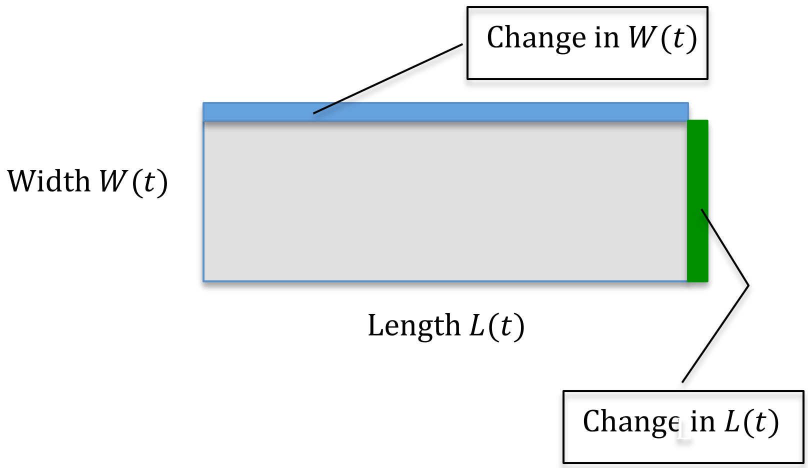
Think of it this way.

Make a rectangle with sides $L(t)$ and $W(t)$



We want the change in the area when the two sides BOTH change.

Make a change in length and width.



$$\frac{d\,A(t)}{dt} = \frac{dL(t)}{dt}\,W(t) + L(t)\,\frac{dW(t)}{dt}$$

This simply says that the change in the area is the sum of the blue rectangle and the green rectangle (plus a tiny bit in the corner).

Example:

Suppose that $y = (x^2 + 7x - 113)(x^2 - 4x + 6)$

Then

$$\frac{dy}{dx}$$

$$= \frac{d(x^2 + 7x - 113)}{dx}(x^2 - 4x + 6) + (x^2 + 7x - 113)\frac{d(x^2 - 4x + 6)}{dx}$$

$$= (2x + 7)(x^2 - 4x + 6) + (x^2 + 7x - 113)(2x - 4)$$

Proof Outline

The definition of the derivative of function f at point a is:

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

If this function is a product f(x) = g(x)p(x) the fraction in the limit is:

$$\lim_{x\to a}\frac{g(x)p(x)-g(a)p(a)}{x-a}$$

Now subtract and add a special term g(a)p(x)

$$\lim_{x\to a}\frac{g(x)p(x)-g(a)p(x)+g(a)p(x)-g(a)p(a)}{x-a}$$

Regroup:

$$\lim_{x\to a}\frac{(g(x)-g(a))p(x)+g(a)(p(x)-p(a))}{x-a}$$

$$\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\lim_{x\to a}p(x)+g(a)\lim_{x\to a}\frac{p(x)-p(a)}{x-a}$$

$$g'(a)\lim_{x\to a}p(x)+g(a)p'(a)$$

We have a proof of the Product Rule as long as we believe that $\lim_{x\to a}p(x)=p(a)$.

This is a property of **continuous functions** and we need function $p(x)$ to be continuous at a. This is true in any case where we use the Product Rule.

$$g'(a)\,p(a)+g(a)p'(a)$$

Summary

1. Derivatives of sums and constants are easy.
2. We use the product rule all the time.
3. The rules required proof; we have outlined the arguments here. You can find more in any textbook or on-line.
4. Using rules correctly is very important.
5. Product Rule is much more exciting when we have transcendental functions to play with.
6. Product Rule is the basis of “Integration by Parts”